Trinity Academy

A Level Further Mathematics Summer Transition Work

Videos

Here are some videos you may find interesting over the summer!

The Chaos Game: https://youtu.be/kbKtFN71Lfs

The Game of Life: https://youtu.be/R9Plq-D1gEk

Calculating a car crash: https://youtu.be/i3D7XYQExt0

The Numberphile YouTube channel has lots of interesting videos on many different aspects of mathematics. Have a look to see what relates to your interests/other subject choices.

Complex Numbers Notes

Read the notes and complete the exercises from the text. This will form part of the discussion of the first lesson.

Complex Numbers

We shall study what is meant by the phrase "complex number".

We shall also study how to find sums, differences and products of complex numbers and how to find complex roots of quadratic equations.

2.1 Origins of Complex Numbers

The history of complex numbers goes back to the ancient Greeks, who were unable to find a real number that was the solution to the equation $x^2 + 1 = 0$.

Clearly, if $x^2 = -1$ then $x = \pm\sqrt{-1}$.

But what real number will square to give a negative answer?

The mathematician Diophantus attempted to solve the problem of finding a right-angled triangle with perimeter 12 units and area 7 square-units. His solution relied on solving the equation $6x^2 - 43x + 84 = 0$. But can this be done?

Cardan posed a similar problem in 1545. Find two numbers α and β such that $\alpha + \beta = 10$ and $\alpha\beta = 40$. This is equivalent to solving the quadratic equation $x^2 - 10x + 40 = 0$. If we use the quadratic formula on Cardan's equation we get:

$$x = \frac{10 \pm \sqrt{100 - 160}}{2}$$

This reduces to: $x = 5 \pm \sqrt{-15}$

It wasn't until the 19th Century that solutions like these were understood.

2.2 The imaginary number i

Mathematically we introduce the symbol i to represent the square root of -1 and remember that it isn't a real number, it is an imaginary number.

This means that we have a new definition.

$$i = \sqrt{-1}$$

Technically we should write $\sqrt{-1} = \pm i$ because we know that a square root can be either positive or negative.

By squaring multiples of i we can now create negative numbers, or put another way, we can now find the square root of a negative number.

Example 2.1

Find the value of $(5i)^2$

Solution: We can first write the problem as a multiplication.

$$(5i)^2 = 5i \times 5i = 25i^2$$

Since $i = \sqrt{-1}$, then $i^2 = -1$. This means that $25i^2 = -25$.

We can also think of this as: $\sqrt{-25} = \pm 5i$

Note carefully the two solutions (one positive, one negative).

Since i is a number we can raise it to different powers.

This gives us the following results:

$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

Subsequent powers then produce cyclic results.

Example 2.2

Simplify $(-4i)^3$

Solution: First write the expression as a product:

$$(-4i)^3 = (-4i) \times (-4i) \times (-4i)$$

Simplifying gives: $(-4i)^3 = -64i^3 = 64i$

Feedback Exercise

Further Pure Maths 1: Exercise 2A (page 19)

2.3 The Complex Conjugate

We can combine imaginary numbers with real numbers.
When we do this we create a complex number.

For example, $2 + 3i$, $-3 - 5i$ and $(\pi + i\sqrt{2})$ are all examples of complex numbers.

Complex numbers all have two parts, a real part and an imaginary part.

We can define complex numbers using algebraic symbols in the following way:

> Any number of the form $p + iq$ where $p, q \in \mathbb{R}$ and $q \neq 0$ is said to be a complex number.
>
> The set of all complex numbers is denoted by $\mathbb{C}$.

Imaginary numbers are really a subset of the complex numbers because all imaginary numbers can be written as $0 + ki$ where $k \in \mathbb{R}$.

We frequently denote complex numbers as $z = p + qi$, where $p, q \in \mathbb{R}$.
The complex conjugate of z is then the complex number with an imaginary part opposite in sign, denoted by z^*.

> This means that if $z = p + qi$ then $z^* = p - qi$

[The importance of the complex conjugate will be seen later.]

2.4 The Arithmetic of Complex Numbers

We can combine complex numbers in similar ways to real numbers.

When we add and subtract complex numbers we deal with the real and imaginary parts separately. In effect we algebraically collect like terms.

Example 2.3

The complex numbers z_1 and z_2 are given by $z_1 = 5 - 3i$ and $z_2 = 4 - 2i$. Find the values of $z_1 + z_2$ and $z_1 - z_2$.

Solution: We can set up equations so that we can collect the like terms.

$$z_1 + z_2 = (5 - 3i) + (4 - 2i) = 9 - 5i$$

Similarly: $$z_1 - z_2 = (5 - 3i) - (4 - 2i) = 1 - i$$

When we multiply complex numbers we have to expand brackets. The simplification also relies on us to remember that $i^2 = -1$.

Example 2.4

If $z_1 = 5 - 3i$ and $z_2 = 4 - 2i$ find the value of $z_1 z_2$.

Solution: We can set up the product as an expression involving brackets. We then expand the brackets and collect like terms:

$$z_1 z_2 = (5 - 3i)(4 - 2i)$$

$$z_1 z_2 = 20 - 10i - 12i + 6i^2$$

$$z_1 z_2 = 14 - 22i$$

There is a special case when we multiply a complex number by its complex conjugate.

Example 2.5

If $z = 5 + 3i$ find the value of $z \times z^*$

Solution: Remember that if $z = 5 + 3i$ then $z^* = 5 - 3i$.

This means that the product gives us:

$$zz^* = (5+3i)(5-3i)$$

$$zz^* = 25 - 15i + 15i - 9i^2 = 34$$

Notice that these two complex numbers, which are conjugate pairs, have a product that is a real number.

This means that complex numbers can be factors of prime numbers. For example:

$$(4+i)(4-i) = 17$$

$$(3+2i)(3-2i) = 13$$

$$(2+3i)(2-3i) = 13$$

So real numbers can have complex number factors that are conjugate pairs.

Feedback Exercise

Further Pure Maths 1: Exercise 2B (page 22)